Goldstein 2.12

Define $I = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, \ddot{q}_i, t)\, dt$ the index i denotes L is a function of a set of $q_i, \dot{q}_i, \ddot{q}_i$ between $i = \{0, 1, \cdots, n\}$.

Introduce variation parameterized by α:

$$q_i, \dot{q}_i, \ddot{q}_i \rightarrow q_i(\alpha), \dot{q}_i(\alpha), \ddot{q}_i(\alpha)$$

Impose vanishing first differential of I with respect to α:

(summation on i implied)

$$0 = \frac{\partial I}{\partial \alpha} = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} \frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial \alpha} + \frac{\partial L}{\partial \ddot{q}_i} \frac{\partial \ddot{q}_i}{\partial \alpha} \right] dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} \frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \ddot{q}_i} \frac{d}{dt} \frac{d}{dt} \frac{\partial q_i}{\partial \alpha} \right] dt$$

Applying integration by parts

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} \frac{\partial q_i}{\partial \alpha} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial \alpha} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}_i} \frac{d}{dt} \frac{\partial q_i}{\partial \alpha} \right] dt + \left. \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial \alpha} \right|_{t_1}^{t_2} + \left. \frac{\partial L}{\partial \ddot{q}_i} \frac{d}{dt} \frac{\partial q_i}{\partial \alpha} \right|_{t_1}^{t_2}$$

we argue that $\left. \frac{\partial q_i}{\partial \alpha} \right|_{t_2} = \left. \frac{\partial q_i}{\partial \alpha} \right|_{t_1} = 0$, because variation by α is subject to the constraint that the end pts are fixed:

$$q(t_1, \alpha_1) = q(t_1, \alpha_2) \text{ for } \alpha_1 \neq \alpha_2 \Rightarrow \left. \frac{\partial q_i}{\partial \alpha} \right|_{t_1} = 0.$$

For $\frac{\partial L}{\partial \ddot{q}_i} \frac{d}{dt} \frac{\partial q_i}{\partial \alpha} \Big|_{t_1}^{t_2}$, observe this is equal to

$$\frac{\partial L}{\partial \ddot{q}_i} \frac{\partial \dot{q}_i}{\partial \alpha} \Big|_{t_1}^{t_2}$$ which shall vanish by same argument.

So we are left with

$$\frac{\partial I}{\partial \alpha} = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} \frac{\partial q_i}{\partial \alpha} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial \alpha} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}_i} \frac{d}{dt} \frac{\partial q_i}{\partial \alpha} \right] dt$$

Applying integration by parts once more yields.

$$\frac{\partial I}{\partial \alpha} = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i} \right] \frac{\partial q_i}{\partial \alpha} \, dt = 0.$$

$\frac{\partial q_i}{\partial \alpha}$ can be made arbitrarily large by shrinking α, so

for this quantity to vanish we need

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i} = 0}$$

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